

STATIONARY SETS IN TOPOLOGICAL AND PARATOPOLOGICAL GROUPS

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ABSTRACT. We show that if a topological or paratopological group G contains a stationary subset of some regular uncountable cardinal, then G contains a subspace which is not collectionwise normal. This statement implies that if a monotonically normal space (in particular, any generalized ordered space) is a paratopological group then the space is hereditarily paracompact.

1. INTRODUCTION

It is proved in [2] that if under certain conditions a group G contains a subset homeomorphic to an uncountable ordinal then G is not hereditarily normal. This statement suggests that a similar statement should be true if one replaces "an uncountable ordinal" by "a stationary subset of an uncountable regular cardinal". The purpose of this paper is just this task. The main result of the paper states that if a topological or paratopological group contains a stationary subset of an uncountable regular cardinal, then the group is not hereditarily collectionwise normal (or, more precisely, contains a subspace which is either not normal or not collectionwise Hausdorff). Using our main result and the Balogh-Rudin generalization [1] of the Engelking-Lutzer theorem [8], we conclude that every monotonically normal topological (or paratopological) group is hereditarily paracompact. This significantly improves an earlier result in [9], namely, by considerably relaxing the hypothesis and significantly strengthening the conclusion.

In notation and terminology we will follow [3]. All spaces under consideration are Tychonov. If X is a generalized ordered space and Y is its subspace then by $[a, b]_Y$ we denote the trace of the segment $[a, b] \subset X$ in Y . The same concerns other types of intervals. If f is a continuous function from X to Y , by \tilde{f} we denote its continuous extension to the Čech-Stone compactifications βX and βY . As usual, G is a *paratopological group* if G is a group, a topological space, and the group binary operation is continuous with respect to the topology of G . Basic facts about ordinals and their stationary subsets can be found, in

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particular, in [7] or any other introductory text on set theory. Ordinals and their subsets, when considered as topological spaces, are endowed with the topology of linear order and the subspace topology, respectively.

2. RESULT

We start with the statement which is a corollary to the classical argument of Katetov [6] (or see Exercise 2.7.15(a) in [3]).

Lemma 2.1. (Follows from [6]) *Suppose that S is a stationary subset of a regular uncountable cardinal κ , $\lambda < \kappa$, $\lambda \in L \subset [0, \lambda]$, λ is a limit point of L , and*

$$K = \{\langle x, y \rangle \in L \times S : (x < \lambda) \text{ or } (x = \lambda \text{ and } y \text{ is isolated in } S)\}.$$

Then $A = \{\langle \lambda, y \rangle \in K\}$ and $B = \{\langle x, y \rangle \in K : y \text{ is limit in } S\}$ are closed and disjoint subsets of K that cannot be separated by disjoint open neighborhoods in K .

Proof. (Natural modification of the argument in [6]) It is clear that A and B are closed and disjoint in K . Let U be any open set in K that contains B . We need to show that A meets the closure of U in K . By stationarity of $\{\alpha \in S : \alpha \text{ is a limit point for } S\}$, for every $x \in L \setminus \{\lambda\}$ we can find $y_x < \kappa$ such that $\{x\} \times [y_x, \kappa)_S \subset U$. Since $|L| < \kappa$, we have $\bar{y} = \sup\{y_x : x \in L \setminus \{\lambda\}\} < \kappa$. Thus, $(L \setminus \{\lambda\}) \times [\bar{y}, \kappa)_S$ is a subset of U , meaning that the closure of U meets A . \square

Lemma 2.2. *Let G be a paratopological group with binary operation \star and let $T \subset G$ be a stationary subset of a regular uncountable cardinal κ . Then it is possible to choose a closed unbounded subset S of the set T , an ordinal λ in T , and a set $L \subset [0, \lambda]_T$ that have the following properties:*

- (1) $\lambda \in L$,
- (2) λ is a limit point of L , and
- (3) the operation \star is injective on $L \times S$.

Proof. We will start with the following claim.

Claim. *For any $\alpha \in T$ there exists $\lambda_\alpha \in T$ such that $\alpha \star y \neq x \star z$ whenever $x, y, z \in [\lambda_\alpha, \kappa)_T$.*

To prove the claim we assume that no such λ_α exists. Then for each $\gamma \in T$ we can find $x_\gamma, y_\gamma, z_\gamma \in T$ with the following properties:

$$\text{P1: } \alpha \star y_\gamma = x_\gamma \star z_\gamma;$$

P2: $x_\gamma, y_\gamma, z_\gamma \in (\gamma, \kappa)_T$;

P3: $x_\gamma, y_\gamma, z_\gamma \in (\max\{x_\beta, y_\beta, z_\beta\}, \kappa)_T$ if $\gamma > \beta$.

By stationarity of T we can find $\bar{\gamma} \in T$ a limit point for T such that

P4: $\bar{\gamma} = \sup\{x_\gamma : \gamma < \bar{\gamma}\} = \sup\{y_\gamma : \gamma < \bar{\gamma}\} = \sup\{z_\gamma : \gamma < \bar{\gamma}\}$

By continuity of \star and P1, we have $\alpha \star \bar{\gamma} = \bar{\gamma} \star \bar{\gamma}$. This equality implies that $\alpha = \bar{\gamma}$, which contradicts P2 and P3 and proves the claim.

By claim we can find an unbounded subset $P \subset T$ with the following property:

(*) : $\lambda_\alpha < \beta$ whenever $\alpha, \beta \in P$ and $\alpha < \beta$.

Now, by stationarity of T , we can find a point $\lambda \in T$ which is a limit point of the set P . Put $S = [\lambda, \kappa)_T$ and $L = \{\alpha \in P : \alpha \leq \lambda\} \cup \{\lambda\}$. By (*) and the property of λ_α 's in the claim conclusion, S and L are as desired. \square

Lemma 2.3. *Suppose that S is a stationary subset of an uncountable regular cardinal κ , $\lambda < \kappa$, $\lambda \in L \subset [0, \lambda]$, λ is a limit point for L , and f is a continuous bijection of $L \times S$ onto its image Z such that $f|_{\{\lambda\} \times S}$ is a homeomorphism. Then Z contains a subspace which is not normal or not collectionwise Hausdorff.*

Proof. We may assume that Z is normal. To simplify our formulas put

$$V_\lambda = \{\lambda\} \times S \text{ and } V_\lambda^\alpha = \{\lambda\} \times (\alpha, \kappa)_S.$$

Claim 1. *If $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in L \times S$ and $f(x_1, y_1), f(x_2, y_2) \in Cl_Z(f(V_\lambda^\alpha))$ for all $\alpha < \kappa$, then $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$.*

To prove the claim, we first observe that any Hausdorff compactification of V_λ has only one point of complete accumulation of V_λ . This follows from the fact that κ is regular and S is stationary in κ . Denote this point in $Cl_{\beta(L \times S)}(V_\lambda)$ by p . Then by the claim assumption we have $\tilde{f}(p) = \tilde{f}(x_1, y_1) = \tilde{f}(x_2, y_2)$, which implies that $f(x_1, y_1) = f(x_2, y_2)$. Since f is injective, we arrive at the conclusion of the claim.

Note that if $f(x, y)$ is not a complete accumulation point of $f(V_\lambda)$ but belongs to the closure of $f(V_\lambda)$ in Z , then $f(x, y)$ belongs to the closure of $f(\{\lambda\} \times [0, \alpha)_S)$ in Z for some $\alpha < \kappa$.

Claim 2. *There exists $\alpha < \kappa$ such that $f(V_\lambda^\alpha)$ is closed in $Z_\alpha = f(L \times (\alpha, \kappa)_S)$.*

To prove the claim, we assume the contrary. Then, by Claim 1, for every $\alpha < \kappa$ we can find a pair $\langle x_\alpha, y_\alpha \rangle \in [L \times (\alpha, \kappa)_S] \setminus V_\lambda^\alpha$ and an element $z_\alpha \in (\alpha, \kappa)$ with the following properties:

- 1: $f(x_\alpha, y_\alpha) \in Cl_{Z_\alpha}(f(\{\lambda\} \times (\alpha, z_\alpha)_S))$,
- 2: $y_\alpha, z_\alpha > \max\{y_\beta, z_\beta, \beta\}$ for ever $\alpha > \beta$.

Since $cf(\kappa) = \kappa > \lambda$, we can also assume that

3: $x_\alpha = x_\beta = x$ for all $\alpha, \beta < \kappa$.

Also, by our choice:

4: $x \neq \lambda$.

By (2) and stationarity of S , we can find an element $y \in S$ that has the following property:

5: $y = \sup\{y_\alpha : \alpha < y\} = \sup\{z_\alpha : \alpha < y\}$.

Applying (1),(3),(5) and continuity of f , we conclude that $f(\lambda, y) = f(x, y)$. This equality and property (4) contradict the fact that f is injective. Claim 2 is proved.

By virtue of Claim 2, we may assume that $f(V_\lambda)$ is closed in Z .

Claim 3. For any $\alpha < \kappa$ there exist $a_\alpha < \lambda$ and $b_\alpha < \kappa$ such that the closure of $f([a_\alpha, \lambda]_L \times [b_\alpha, \kappa)_S)$ in Z misses the closure of $f([a_\alpha, \lambda]_L \times \{\alpha\})$ in Z .

Since $f(V_\lambda)$ is closed in Z , the set V_λ^α is closed in V_λ , and $f|_{V_\lambda}$ is a homeomorphism, we conclude the set $f(V_\lambda^\alpha)$ is closed in Z . Since f is one-to-one, $f(\lambda, \alpha) \notin f(V_\lambda^\alpha)$. By regularity of Z , we can find open sets U and W in Z that contain $f(V_\lambda^\alpha)$ and $f(\lambda, \alpha)$, respectively, such that $Cl_Z(U) \cap Cl_Z(W) = \emptyset$. Since S is stationary in κ and $\lambda < cf(\kappa)$, we can find an element $b_\alpha < \kappa$ such that $\{\lambda\} \times [b_\alpha, \kappa)_S$ has a rectangular neighborhood $[a'_\alpha, \lambda]_L \times [b_\alpha, \kappa)_S$ contained in $f^{-1}(U)$. Since λ is a limit point of L , we can find $a''_\alpha < \lambda$ such that $[a''_\alpha, \lambda] \times \{\alpha\} \subset f^{-1}(W)$. Clearly, $a_\alpha = \max\{a'_\alpha, a''_\alpha\}$ and b_α are as desired. The claim is proved.

For the remaining argument put

$$P_\alpha = L \times [0, \alpha]_S \text{ and as above } V_\lambda^\alpha = \{\lambda\} \times (\alpha, \kappa)_S.$$

The rest of the proof is handling of the following two cases.

Case 1. The assumption for this case is: "There exists α such that $Cl_Z(f(P_\alpha))$ meets $f(V_\lambda^\gamma)$ for every $\gamma > \alpha$." To handle this case, for every $\gamma > \alpha$, fix $\langle \lambda, y_\gamma \rangle \in V_\lambda^\gamma$ such that $f(\lambda, y_\gamma)$ belongs to $Cl_Z(f(P_\alpha))$. Thus, there exists a κ -sized $S' \subset S$ which is discrete in itself and $f(\{\lambda\} \times S')$ is in the closure of $f(P_\alpha)$ in Z . Since $f(V_\lambda)$ is closed in Z and $f|_{V_\lambda}$ is a homeomorphism, we conclude that the set $f(\{\lambda\} \times S')$ is of cardinality κ and is closed and discrete in $f(\{\lambda\} \times S') \cup f(P_\alpha)$. Since the closure of $f(P_\alpha)$ in Z has density at most $|\lambda \times \alpha| < \kappa$, we conclude that $Cl_Z(f(P_\alpha))$ is not hereditarily collectionwise Hausdorff, which completes Case 1.

Case 2. The assumption for this case is: "*Case 1 does not take place.*" To handle this case we will find K, A , and B as in Lemma 2.1. Since Case 1 does not take place we can find a closed unbounded S' in S such that

P1: $Cl_Z(f(P_\alpha)) \cap f(V_\lambda^\beta) = \emptyset$ for any $\alpha, \beta \in S'$, where $\alpha < \beta$

For each $\alpha \in S'$, fix a_α and b_α as in Claim 2. There exists S'' closed and unbounded in S' such that

P2: $\beta > b_\alpha$ for any $\alpha, \beta \in S''$ with $\alpha < \beta$.

In other words,

P3: $Cl_Z(f([a_\alpha, \lambda]_L \times \{\alpha\}))$ misses $Cl_Z(f([a_\alpha, \lambda]_L \times [\alpha + 1, \kappa)_{S''}))$.

By Claim 3 there exist a κ -sized subset $S''' \subset S''$ and an ordinal $a < \lambda$ such that $a_\alpha = a$ for every $\alpha \in S'''$. Let $\hat{S} = Cl_S(S''')$. Put $Y = [a, \lambda]_L \times \hat{S}$. By P1 and P3 we have

P4: $f(H_\alpha \cap Y)$ is clopen in $f(Y)$ for any $\alpha \in \hat{S}$.

We are ready to define K, A , and B as in Lemma 2.1. Put $K = ([a, \lambda]_L \times \hat{S}) \cup \{\langle \lambda, y \rangle : y \text{ is isolated in } \hat{S}\}$. We have $A = \{\langle \lambda, y \rangle \in K\}$ and $B = \{\langle x, y \rangle \in K : y \text{ is limit in } \hat{S}\}$ are closed and disjoint subsets of K that cannot be functionally separated as in Lemma 2.1. By P4, the set $Cl_{f(K)}(f(B))$ misses $f(A)$. Since $f(V_\lambda)$ is closed in Z and f is injective, we conclude that $f(A)$ is closed in $f(K)$. Since A and B cannot be functionally separated in K , we conclude that $Cl_{f(K)}(f(B))$ and $f(A)$ cannot be functionally separated in $f(K)$ either. Since the closures of the images of A and B in $f(K)$ are disjoint we conclude that $f(K)$ is not normal, which completes the second case and proves our statement. \square

Now we are ready to derive our main results.

Theorem 2.4. *Let G be a paratopological group. If G contains a stationary subspace of an uncountable regular cardinal, then the following are true:*

- (1) G is not hereditarily normal or not hereditarily collectionwise Hausdorff;
- (2) G is not hereditarily collectionwise normal.

Proof. Clearly (2) follows from (1). Let us show that (1) holds. Let S, L , and λ be as in the conclusion of Lemma 2.2. Since multiplication by an element of G is an automorphism, we conclude that $\star|_{\{\lambda\} \times S}$ is a homeomorphism. Therefore, $f = \star$, S, L , and λ satisfy the hypothesis of Lemma 2.3. The conclusion of Lemma 2.3 completes the proof. \square

For our further discussion we would like to state the version of Jones result [5]: Suppose that X has density κ and has a closed discrete subspace of cardinality τ . If $2^\kappa < 2^\tau$ then X is not normal. Observe that in Lemma 2.3, the first case

that leads to finding a subspace which is not collectionwise Hausdorff and has a closed discrete subspace of cardinality greater than the density. Therefore if one assumes the Generalized Continuum Hypothesis and uses the mentioned Jones Lemma, then both cases of the proof of Lemma 2.3 lead to a subspace which is not normal. We can state this observation as follows.

Theorem 2.5. *Assume the Generalized Continuum Hypothesis. Let G be a paratopological group. If G contains a stationary subspace of an uncountable regular cardinal, then G is not hereditarily normal.*

It is of course natural to wonder if a paratopological group containing a stationary subset of an uncountable regular cardinal is not hereditarily normal without any set-theoretic assumptions.

For our next corollary we need the fact that every monotonically normal space is hereditarily collectionwise normal [4, Theorem 3.1] and hereditarily monotonically normal (which follows from the definition).

Theorem 2.6. *Let G be a paratopological group. If G is monotonically normal, then G is hereditarily paracompact.*

Proof. The Balogh-Rudin generalization [1] of the Engelking-Lutzer theorem states that if a monotonically normal space is not paracompact then it contains a stationary subset of an uncountable regular cardinal as a closed subspace. Now assume G is not hereditarily paracompact. Since monotone normality property is inherited by all subspaces it follows from the Balogh-Rudin theorem that G contains a stationary subset of an uncountable regular cardinal. By (2) of Theorem 2.4, G is not hereditarily collectionwise normal. However monotone normality implies that every subspace of G is collectionwise normal. This contradiction completes the proof. \square

Since every subspace of a linearly ordered space is monotonically normal we have the following statement.

Corollary 2.7. *Let G be a paratopological group. If G is a subspace of a linearly ordered space then G is hereditarily paracompact.*

Corollary 2.7 significantly improves an earlier result in [9] in which the hypothesis has one more additional condition, namely, that the group binary operation preserves the order, and the conclusion is paracompactness of the entire space only.

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